Cavity dispersion equation $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$: a note on its origin

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A simple derivation of the cavity dispersion equation for high-gain pulsed lasers, $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$, is provided by using Dirac's notation for probability amplitudes as applied to the analysis of dispersive cavities.

Introduction

The cavity dispersion expression $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$ is widely used to estimate the dispersive linewidth in high-gain pulsed lasers incorporating dispersive optical elements.[1–5] In this equation, $\Delta\theta$ represents the beam divergence and $(\partial\theta/\partial\lambda)$ is the total intracavity dispersion. In practice, this dispersion can be high in multiple-prism grating arrangements since the dispersion of the grating is multiplied by the total beam expansion provided by the multiple-prism array that can be in the 100–200 range.[5] In this regard, tunable pulsed laser oscillators utilizing dispersive optics have been demonstrated to oscillate in a single-longitudinal mode.[6–8]

To physicists working on pulsed lasers, $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$ is a familiar statement with clear and direct experimental relevance. Although this expression can be considered as a purely mathematical statement, historically it can be identified with developments in classical optics. For example, a literature survey indicates that this equation has been derived in the context of geometrical optics and has been applied extensively in dispersive spectrometers.[9–12] In this regard, it appears that laser physicists have adopted this well-known relation from geometrical optics.

In this paper we show that $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$ can be established by using Dirac's bra-ket formalism[13] in the analysis of dispersive cavities. In addition, the simple approach described here may offer further elucidation on the contribution of intracavity dispersion to linewidth narrowing in high-gain dispersive oscillators.

Background

Consider a dispersive resonator incorporating an output coupler, a gain medium, a multiple-prism assembly, and a grating (Fig. 1). The multiple-prism chain is composed of a number of prisms in additive configuration, thus providing a significantly high dispersion and consequently a highly selective frequency transmission function. The overall dispersion for orthogonal beam exit is given by[5]

$$(\partial\theta/\partial\lambda) = M(\partial\theta/\partial\lambda)_G + 2\sum_{m=1}^{r}(\pm 1)\left(\prod_{j=1}^{m} k_{1,j}\right)\tan\psi_{1,m}(\partial n_m/\partial\lambda), \quad (1)$$

where r is the total number of prisms, $(\partial\theta/\partial\lambda)_G$ is the grating dispersion, $(\partial n_m/\partial\lambda)$ is typical of the prism material, and

$$M = \prod_{m=1}^{r} k_{1,m} \quad (2)$$

is the total beam magnification, where $k_{1,m} = (\cos\psi_{1,m}/\cos\phi_{1,m})$ is the individual beam-expansion factor. Here, $\phi_{1,m}$ is the incident angle at the mth prism, and $\psi_{1,m}$ is the corresponding refractive angle.

In addition to frequency selectivity, the multiple-prism grating assembly allows only the transmission of highly p-polarized radiation. The cumulative transmission losses at the incident surface of the mth prism are given by

$$L_{1,m} = L_{2,(m-1)} + [1 - L_{2,(m-1)}]R_{1,m}, \quad (3)$$

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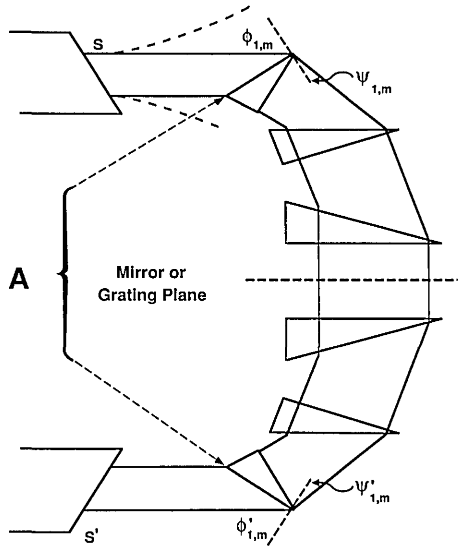

Fig. 1. Schematic of the unfolded dispersive assembly of the cavity integrated by a multiple-prism expander and grating. Single-pass radiation exits the gain medium at s and is incident at an angle $\phi_{1,m}$ at the prismatic expander. The dispersive assembly A allows emission within a resonant narrow-bandwidth range to exit the prisms at the angle $\phi_{1,m}'$. Radiation that exits at this angle proceeds to return to the narrow-band amplification beam axis at the gain medium (s').

where

$$R_{1,m} = \tan^2(\phi_{1,m} - \psi_{1,m})/\tan^2(\phi_{1,m} + \psi_{1,m}) \quad (4)$$

is the usual Fresnel equation for p polarization.[10] Equations (1)–(4) are provided as background information.

Intuitive Approach

Here we describe an intuitive approach to establishing a correspondence between a probabilistic description of intracavity propagation, in a dispersive cavity, and $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$. The approach applies the Dirac formalism[13] and follows a basic and simple probabilistic style similar to that outlined by Feynman *et al.*[14]

Let us assume, on a single-pass perspective, that unpolarized and highly divergent amplified spontaneous emission of a broad frequency range leaves the gain medium s toward the dispersive assembly A (see Fig. 1). Transmission at the dispersive assembly is restricted only to radiation incident at the specific angle $\phi_{1,m}$ that satisfies resonant frequency conditions. Hence, if s' is the gain axis for narrow-linewidth oscillation, then the probability amplitude for resonant narrow-linewidth double-pass amplification can be written as

$$\langle s'|\mathrm{A}|s\rangle = \sum_{\phi_{1,m}} \langle s'|\phi_{1,m}'\rangle \langle \phi_{1,m}'|\mathrm{A}|\phi_{1,m}\rangle\langle\phi_{1,m}|s\rangle. \quad (5)$$

Since $\phi_{1,m}$ is a unique angle of incidence on the gain axis at the multiple-prism expander that is necessary to induce diffraction at the grating followed by return passage, then the probability for the transmission of monochromatic radiation is given by

$$|\langle s'|\mathrm{A}|s\rangle|^2 = |\langle s'|\phi_{1,m}'\rangle|^2|\langle\phi_{1,m}'|\mathrm{A}|\phi_{1,m}\rangle|^2|\langle\phi_{1,m}|s\rangle|^2. \quad (6)$$

Now, the probability for narrow-linewidth propagation of the photon flux leaving s toward A is inversely proportional to its divergence $\Delta\theta$ so that

$$|\langle\phi_{1,m}|s\rangle|^2 = \kappa_1(1/\Delta\theta). \quad (7)$$

Here we should note that $\Delta\theta$ is always finite, and its minimum value is its well-known diffraction limit $\Delta\theta_D = \lambda/(\pi w)$, where w is the beam waist.[15] Values for $\Delta\theta$ can be derived classically by using propagation matrices[5] or by using the uncertainty principle.[16]

Once the photon flux arrives at the dispersive chain, the return of resonant narrow-linewidth emission depends on the dispersion

$$|\langle\phi_{1,m}'|\mathrm{A}|\phi_{1,m}\rangle|^2 = \kappa_2(\partial\theta/\partial\lambda). \quad (8)$$

For highly selective resonant radiation that returns at $\phi_{1,m}'$, the probability to undergo amplification at s' is high, thus

$$|\langle s'|\phi_{1,m}'\rangle|^2 \approx 1. \quad (9)$$

Since the overall probability for resonant amplification is inversely proportional to the wavelength spread of the emission

$$|\langle s'|\mathrm{A}|s\rangle|^2 = \kappa_3(1/\Delta\lambda), \quad (10)$$

then substitution of Eqs. (7), (8), and (10) and approximation (9) into Eq. (6) yields

$$\Delta\lambda = \Delta\theta(\partial\theta/\partial\lambda)^{-1} \quad (11)$$

for the special case[17] of $\kappa_3 \approx \kappa_1\kappa_2$.

Following consecutive return passes, $\Delta\theta$ should be expected to decrease toward its diffraction limit. This, in turn, allows for a refinement in $\Delta\lambda$, as is well known in long-pulse dye lasers.[4,5] Measurements on the consecutive refinement in the values of $|\langle\phi_{1,m}|s\rangle|^2$ and $|\langle s'|\mathrm{A}|s\rangle|^2$ could be used to determine the influence of multipass dispersive effects on $\Delta\lambda$. In turn, this should provide information that is necessary to develop a multipass linewidth theory including the dynamics of the gain medium.

Derivation by Using the Generalized Interference Equation

Let us consider the generalized case of a transmission grating being illuminated by a dispersionless beam expander as illustrated in Fig. 2. The probability amplitude for a photon to go from the exit surface of the beam expander (s) to a total reflector (x) via an array of N slits (j) can be written as[13]

$$\langle x|s\rangle = \sum_{j=1}^{N} \langle x|j\rangle\langle j|s\rangle. \quad (12)$$

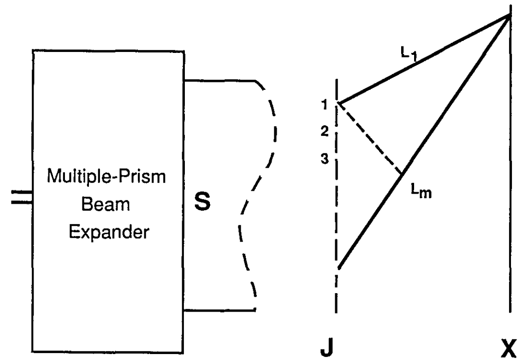

Fig. 2. Expanded beam (S) illuminates a large number of slits of a transmission grating (J). The photons then proceed to reach X, where interference is observed. Here, the beam expander is assumed to be of a dispersionless design.[5]

The probability amplitudes can be expressed in the form of $\langle x|j\rangle = \Psi(r_{j,x})\exp(-i\phi_j)$ and $\langle j|s\rangle = \Psi(r_{s,j})\exp(-i\theta_j)$, where $\Psi(r_{j,x})$ and $\Psi(r_{s,j})$ are appropriate diffraction wave functions.[14] Thus, the generalized probability for one-dimensional N-slit interference can be written as[18]

$$|\langle x|s\rangle|^2 = \sum_{j=1}^{N} \Psi(r_j)^2 + 2\sum_{j=1}^{N} \Psi(r_j) \times \left[\sum_{m=j+1}^{N} \Psi(r_m)\cos(\Omega_m - \Omega_j)\right], \quad (13)$$

where $\Psi(r_j) = \Psi(r_{j,x})\Psi(r_{s,j})$ and $\Omega_j = (\theta_j + \phi_j)$. This is a generalized interference equation that has been successfully applied to predict accurately interference and/or diffraction phenomena arising from N-slit gratings.[18] The beauty of this equation is its simplicity and general applicability. In this regard, Eq. (13) has been shown to be applicable in the near and/or far field and can also be used to predict transverse mode structure that is due to intracavity apertures.

For uniform illumination of the grating

$$\langle 1|s\rangle = \langle 2|s\rangle = \langle 3|s\rangle = \ldots = \langle N|s\rangle = \Psi(r_{s,j})\exp(-i\theta_j) = 1,$$

and the interference equation simplifies to

$$|\langle x|s\rangle|^2 = \sum_{j=1}^{N} \Psi(r_{j,x})^2 + 2\sum_{j=1}^{N} \Psi(r_{j,x}) \times \left[\sum_{m=j+1}^{N} \Psi(r_{m,x})\cos(\phi_m - \phi_j)\right]. \quad (14)$$

The interference term in Eq. (14) is

$$\cos(\phi_m - \phi_j) = \cos \mathbf{k}\cdot\mathbf{r} = \cos \mathbf{k}|L_m - L_{m-1}|,$$

where $|L_m - L_{m-1}|$ is the exact optical path difference that can be approximated as[10]

$$|L_m - L_{m-1}| = d_m \sin\theta_m.$$

For a grating with uniform slit separation the maxima is given by $kd \sin\theta = n\pi$. Using $k = (2\pi)/\lambda$ one can write the well-known grating equation in Littrow configuration:

$$n\lambda = 2d\sin\theta. \quad (15)$$

For two wavelengths separated by a small difference $\Delta\lambda$ so that $\theta_1 \approx \theta_2\ (= \theta)$, we can write

$$n\Delta\lambda \approx 2d\Delta\theta\{1 - (3\theta^2/3!) + (5\theta^4/5!)\ldots\}. \quad (16)$$

Differentiating Eq. (15) and substituting the result into approximation (16) yields

$$\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}\{1 - (\theta^2/2!) + (\theta^4/4!)\ldots\}(1/\cos\theta). \quad (17)$$

This expression reduces to $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$.

A further avenue to establishing expression Eq. (11) is to use the time-independent component of the wave equation $\Psi = \Psi_0 \exp(ikx)$. Following differentiation and for a small wavelength difference, $\Delta\lambda \approx 2\pi\Delta\Psi(\partial\Psi/\partial x)^{-1}$. Then, assuming a Gaussian propagation behavior for Ψ and using $\Delta p\Delta x \approx \hbar$, we can establish expression (11).

Conclusion

We have established $\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1}$ through the application of the Dirac formalism to the description of propagation in dispersive cavities. This was accomplished by using a return-pass description of intracavity propagation in a multiple-prism grating cavity and also by using the generalized interference equation.

Although the approach is by no means rigorous and contains various practical approximations, it illustrates that this widely used expression in the field of pulsed lasers is compatible with the application of the Dirac formalism to dispersive cavities. From a practical perspective, the elements for an iterative measurement and computational approach to determine the influence of multipass dispersive effects on $\Delta\lambda$ have been outlined.

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